TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 340, Number 1, November 1993

THE NEF VALUE AND DEFECT OF HOMOGENEOUS LINE BUNDLES

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ABSTRACT. Formulas for the nef value of a homogeneous line bundle are derived and applied to the classification of homogeneous spaces with positive defect and to the classification of complete homogeneous real hypersurfaces of projective space.

Let X be a smooth projective variety imbedded in $\mathbb{P}^N$ by the sections of some very ample line bundle L. If the canonical bundle K_X is not numerically effective (nef), then there is a smallest rational number $\tau = \tau(X, L)$ called the nef value of (X, L) such that $K_X \otimes L^\tau$ is nef. The map $\psi: X \to Y$ defined by the sections of some power of $K_X \otimes L^\tau$ is called the nef value morphism. In this paper a general formula is derived for the nef value of L when X is a homogeneous space equivariantly imbedded in $\mathbb{P}^N$ by the sections of L, see Theorem 2.2. It is then an easy matter to tabulate the exact values for $\tau(X, L)$ when $\operatorname{Pic}(X) \cong \mathbb{Z}$, see Corollary 2.4.

As is shown in [2, 3], there is a connection between the nef value, $\tau(X, L)$, and the codimension of the variety $X' \subset \mathbb{P}^N$ of hyperplanes tangent to X, known as the dual or discriminant variety of X. The defect of (X, L) is defined to be $\operatorname{def}(X, L) = \operatorname{codim} X' - 1$. Most smooth varieties have defect 0. If $\operatorname{def}(X, L) > 0$, then the defect is determined by the nef value [2],

$$\operatorname{def}(X, L) = 2(\tau(X, L) - 1) - \dim X.$$

Moreover, if Z is a general fiber of the nef value morphism $\psi: X \to Y$, then $\operatorname{def}(X, L) = \operatorname{def}(Z, L_Z) - \dim Y$ and $\operatorname{Pic}(Z) \cong \mathbb{Z}$. If the defect of X is greater than 1, then a smooth hyperplane section of X also has positive defect, see [7]. Up to such hyperplane sections and fibrations, the only known examples of smooth varieties with positive defect k are linear projective spaces, $\mathbb{P}^n$, $k = n$, the Plücker imbedding of the Grassmann variety, $\operatorname{Gr}(2, 2m+1)$, $k = 2$, and the 10-dimensional spinor variety S_4 in $\mathbb{P}^{15}$, $k = 4$. These last examples are all homogeneous spaces. In fact, they are the only homogeneous projective varieties with $\operatorname{def}(X, L) > 0$, along with products $X_1 \times X_2$ built from them satisfying $\operatorname{def}(X_1) - \dim X_2 > 0$, see [11]. The proof of the classification given in [11] is difficult and proceeds through many cases based on the type of the group. In §4 a simple proof is given based on the above relationship between the defect and the nef value. Only a few special cases arise which are handled

Received by the editors August 19, 1991.
1991 *Mathematics Subject Classification*. Primary 14M17; Secondary 32M10.

by determining the fiber dimension of the duality map $\phi: \mathbb{P}(N_X^*(1)) \to \mathbb{P}^N$. A general method for calculating such fiber dimensions is presented in Theorem 3.7 which reduces the problem for these cases to computing the rank of a matrix constructed from structure constants and weights.

Finally, a list of the self-dual homogeneous spaces is derived, see Corollary 4.3. Up to Hartshorne's Conjecture, this is the same list obtained without the assumption of homogeneity, see [6]. Curiously, these spaces also appear in the classification of homogeneous real hypersurfaces in $\mathbb{P}^N$, see [1, 15]. This connection is explained in Corollary 4.4.

I would like to thank Andrew Sommese for his many helpful discussions, particularly relating to Proposition 3.4.

1. Preliminaries

In this section we organize some facts about homogeneous spaces. General references are [4, 9]. We assume throughout that we are working over the complex numbers $\mathbb{C}$.

Let G be a simply-connected semisimple complex Lie group. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let Λ denote the character group of T which we write additively and refer to as the weights of G. Let $\Phi \subset \Lambda$ denote the roots of G relative to T. To every root $\alpha \in \Phi$ there corresponds a 1-dimensional unipotent subgroup $U_\alpha \cong \mathbb{C}$ called the root group of α. We define the negative roots, Φ^-, to be those α such that $U_\alpha \subset B$. In particular, B is generated by T and the negative root groups U_α, $\alpha \in \Phi^-$. The positive roots are $\Phi^+ = \Phi\backslash\Phi^- = -\Phi^-$. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset \Phi^+$ denote the subset of simple roots, $l = \dim T = \operatorname{rank} G$, so that each root α can be expressed uniquely as an integral combination $\alpha = \sum_i n_i(\alpha)\alpha_i$ with either all $n_i(\alpha) \geq 0$ $(\alpha \in \Phi^+)$ or all $n_i(\alpha) \leq 0$ $(\alpha \in \Phi^-)$.

The weights are generated as a $\mathbb{Z}$-module by the fundamental weights $\lambda_1, \ldots, \lambda_l$ dual to the simple roots under the Killing form: $(\lambda_i, \alpha_j) = \delta_{ij}(\alpha_j, \alpha_j)/2$. A weight $\lambda = \sum_i m_i(\lambda)\lambda_i$ is dominant if all $m_i(\lambda) \geq 0$ (equivalently, $(\lambda, \alpha) \geq 0$ for all positive roots $\alpha \in \Phi^+$). We denote the dominant weights by Λ^+. A weight λ is regular if $(\lambda, \alpha) \neq 0$ for all positive roots $\alpha \in \Phi^+$. There is also a partial ordering on Λ: $\lambda > \mu$ iff $\lambda - \mu$ is 0 or a sum of positive roots.

A parabolic subgroup P of G is a subgroup that contains a Borel subgroup. Up to conjugacy, we may assume that P contains B and therefore is determined by a subset of simple roots $\Psi \subset \Delta$. Let $\langle\Psi\rangle$ denote the positive roots of G that are linear combinations of the roots in Ψ. The subgroup P is generated by B and all the root groups U_α for $\alpha \in \langle\Psi\rangle$. If $X = G/P$, then we refer to the roots of G that are not roots of P as the roots of X, denoted R_X. Clearly, $R_X = \langle\Delta\rangle\backslash\langle\Psi\rangle$. We denote by $\Delta_X = \Delta\backslash\Psi$ the set of simple roots of X. A parabolic subgroup P is maximal if it is generated by a maximal subset of simple roots $\Psi = \Delta\backslash\{\alpha_i\}$ so that $\Delta_X = \{\alpha_i\}$.

If P acts linearly on a vector space E, then we define the twisted product $G \times_P E$ to be the quotient of $G \times E$ by the diagonal action of P: $p \cdot (g, z) = (gp^{-1}, p \cdot z)$. We represent an equivalence class in $G \times_P E$ by a pair $[g, z]$, remembering that $[gp, z] = [g, p \cdot z]$ for $p \in P$. Projection onto the first coordinate induces a map $\pi: G \times_P E \to X = G/P$, $\pi([g, z]) = gP$, which realizes the twisted product as a vector bundle on X with fiber E. Any vector bundle on X which is invariant under G can be realized in this way. Since any line bundle L on X is invariant under G, L is isomorphic to $G \times_P \mathbb{C}$ where P acts on $\mathbb{C}$ by a character $\lambda\colon P \to \mathbb{C}^*$. The character λ also defines by restriction a character on a maximal torus T and therefore a weight of G. To be a character on P the weight λ must be orthogonal to the positive roots of P. Therefore, λ must be an integral combination of the fundamental weights $\mu_1, \dots, \mu_t$ dual to the simple roots in Δ_X. We call $\mu_1, \dots, \mu_t$ the fundamental weights of X and the sublattice $\Lambda_X \subset \Lambda$ they generate the weights of X.

Every dominant weight $\lambda \in \Lambda^+$ defines a character on B and therefore determines a line bundle $L = G \times_B \mathbb{C}$ on $X = G/B$. The vector space of sections $V = H^0(X, L)$ is an irreducible representation of G and every irreducible representation of G is obtained in this way. If $\Psi_\lambda = \{\alpha \in \Delta | (\lambda, \alpha) = 0\}$ then Ψ_λ defines a parabolic subgroup P_λ to which λ can be extended and the corresponding line bundle L on G/P_λ is very ample.

The following proposition summarizes some well-known facts about line bundles on homogeneous spaces.

1.1 **Proposition.** *Let $X = G/P$ where G is a semisimple complex Lie group and P is a parabolic subgroup. Let $\mu_1, \dots, \mu_t$ be the fundamental weights of X and let L be a line bundle on X defined by $\lambda = \sum_{j=1}^t m_j(\lambda)\mu_j \in \Lambda_X$. Then*

(1) *$X = X_1 \times \cdots \times X_s$ where $X_i = G_i/P_i$, G_i is a simple complex Lie group, and P_i is a parabolic subgroup of G_i, $i = 1, \dots, s$.*

(2) *$L = \mathrm{pr}_1^* L_1 \otimes \cdots \otimes \mathrm{pr}_s^* L_s$ where L_i is a line bundle on X_i, $i = 1, \dots, s$.*

(3) $\mathrm{Pic}(X) \cong \Lambda_X$. *In particular,* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *iff P is a maximal parabolic subgroup of G.*

(4) *L is numerically effective (nef) iff λ is dominant, i.e., $m_j(\lambda) \geq 0$ for all $1 \leq j \leq t$.*

(5) *L is ample iff it is very ample iff λ is a regular dominant weight, i.e., $m_j(\lambda) > 0$ for all $1 \leq j \leq t$.*

2. The nef value

Let X be a smooth projective variety.

2.1 **Definition** [2, 0.10]. If the canonical bundle K_X is not nef, then for any very ample line bundle L on X there is a smallest rational number $\tau = \tau(X, L)$ called the nef value of (X, L) such that $K_X \otimes L^\tau$ is nef. Moreover, τ is determined by the condition that $K_X \otimes L^\tau$ is nef but not ample. The map $\psi\colon X \to Y$ defined by the sections of some power of $K_X \otimes L^\tau$ is called the nef value morphism.

We first show how to compute the nef value of an ample line bundle on a homogeneous space $X = G/P$ using root data.

2.2 **Theorem.** *Let $X = G/P$ where G is a semisimple complex Lie group and P is a parabolic subgroup defined by $\Psi \subset \Delta$. Let λ be the weight of the anticanonical bundle $K_X^* = \bigwedge^n T_X$, $n = \dim X = |R_X|$. Then λ is regular and dominant and given by $\lambda = \sum_{\alpha \in R_X} \alpha$. In particular, K_X is not nef. The nef value of an ample line bundle L on X defined by a weight $\mu = \sum_{j=1}^t m_j(\mu)\mu_j \in \Lambda_X^+$ is given by*

$$\tau(X, L) = \max_{1\le j\le t} \frac{m_j(\lambda)}{m_j(\mu)} = \max_{\beta\in\Delta_X} \frac{(\lambda, \beta)}{(\mu, \beta)}.$$

The nef value morphism $\psi: X \to Y$ *is a homogeneous fiber bundle* $G/P \to G/P_1$ *where* P_1 *is the parabolic subgroup defined by* $\Psi \cup \{\alpha_j | j \in J\}$ *and* J *is the set of indexes for which the above maximum occurs.*

Proof. The tangent space at the identity coset in X is isomorphic to the quotient of Lie algebras $\mathfrak{g}/\mathfrak{p}$ on which P acts via the adjoint representation on $\mathfrak{g}$ projected to the quotient. The weights of this representation therefore consist of the positive roots of G which are not in the subgroup P, namely R_X. The nth exterior power of this representation reduces to a 1-dimensional space whose weight λ is the sum of these roots.

Let $\tau = \tau(X, L)$. The weight of $K_X \otimes L^\tau$ is

$$\tau\mu - \lambda = \sum_{j=1}^{t} (\tau m_j(\mu) - m_j(\lambda))\mu_j.$$

Now, $K_X \otimes L^\tau$ is nef but not ample iff $\tau m_j(\mu) - m_j(\lambda) \ge 0$ for all $1 \le j \le t$ and at least one of these coefficients $= 0$, see Proposition 1.1. This gives the first formula for τ. The second follows from $m_j(\nu) = 2(\nu, \beta_j)/(\beta_j, \beta_j)$ for any weight ν and simple root β_j.

The set of simple roots orthogonal to $\tau\mu - \lambda$ is clearly $\Psi \cup \{\alpha_j | j \in J\}$. The character $\tau\mu - \lambda$ therefore extends to the parabolic subgroup P_1 and the corresponding map given by sections is $G/P \to G/P_1$. □

When P is a maximal parabolic subgroup of G the formula of Theorem 2.2 can be refined. Recall that we write a positive root α as a linear combination of simple roots, $\alpha = \sum_{j=1}^{l} n_j(\alpha)\alpha_j$.

2.3 **Corollary.** *Let* $X = G/P$ *where* G *is a simple complex Lie group and* $P = P_i$ *is a maximal parabolic subgroup, so that* $\Delta_X = \{\alpha_i\}$. *Let* L *be the ample generator of the line bundles on* X *defined by the dual fundamental weight* λ_i. *Then* $R_X = \{\alpha \in \Phi^+ | n_i(\alpha) > 0\}$ *and*

$$\tau(X, L) = \frac{(\alpha_i, \alpha_i)}{2(\lambda_i, \lambda_i)} \sum_{\alpha\in R_X} n_i(\alpha).$$

Proof. The roots of X are those α that are not linear combinations of simple roots α_j, $j \ne i$. Thus, R_X consists of those roots whose α_i-coefficient is positive. If λ is the weight of K_X^* then clearly $\lambda = \tau(X, L)\lambda_i$. Since $\lambda = \sum_{\alpha\in R_X} \alpha$, we can use the Killing form to extract $\tau(X, L)$: $(\lambda_i, \lambda_i)\tau(X, L) = \sum_{\alpha\in R_X} (\alpha, \lambda_i) = \sum_{\alpha\in R_X} \sum_j n_j(\alpha)(\alpha_j, \lambda_i) = \sum_{\alpha\in R_X} n_i(\alpha)(\alpha_i, \alpha_i)/2$. □

For each type of simple Lie group G, it is straightforward to list the positive roots and for each maximal parabolic subgroup P to determine the roots of G/P. Plugging these lists into the above formula yields the following table of nef values. We use the notation A_l, B_l, etc., to denote a simply-connected simple complex Lie group of type a_l, b_l, etc., and rank l. We assume the simple roots are ordered according to [16] and denote by P_i the maximal parabolic subgroup generated by $\Psi = \Delta\backslash\{\alpha_i\}$, $1 \le i \le l$.

2.4 **Corollary.** *Let* $X = G/P$ *where* G *is a simple complex Lie group and* $P = P_i$ *is a maximal parabolic subgroup. Let* $n = \dim X$ *and let* L *be the ample generator for the line bundles on* X. *Then the nef value* $\tau = \tau(X, L)$ *is as given below.*

(1) $G = A_l$:

i	$1, \ldots, l$
n	$i(l+1-i)$
τ	$l+1$

(2) $G = B_l$:

i	$1, \ldots, l-1$	l
n	$i(4l+1-3i)/2$	$l(l+1)/2$
τ	$2l-i$	$2l$

(3) $G = C_l$:

i	$1, \ldots, l-1$
n	$i(4l+1-3i)/2$
τ	$2l-i+1$

(4) $G = D_l$:

i	$1, \ldots, l-2$	$l-1, l$
n	$i(4l-1-3i)/2$	$l(l-1)/2$
τ	$2l-i-1$	$2l-2$

(5) $G = E_6$:

i	1, 5	2, 4	3	6
n	16	25	29	21
τ	12	9	7	11

(6) $G = E_7$:

i	1	2	3	4	5	6	7
n	27	42	50	53	47	33	42
τ	18	13	10	8	11	17	14

(7) $G = E_8$:

i	1	2	3	4	5	6	7	8
n	57	83	97	104	106	98	78	92
τ	29	19	14	11	9	13	23	17

(8) $G = F_4$:

i	1	2	3	4
n	15	20	20	15
τ	8	5	7	11

(9) $G = G_2$:

i	1	2
n	5	5
τ	5	3

3. The defect

References for this material can be found in [2, 6, 7, 10]. The dual variety of a projective variety X is defined as follows. Let $L = \mathscr{O}_X(1)$ be a very ample line bundle on X and let $V = H^0(X, L)$ so that X is naturally imbedded in $\mathbb{P}^N = \mathbb{P}(V^*)$. Let N_X be the normal bundle of X defined by

$$0 \to T_X \to T_{\mathbb{P}^N}|_X \to N_X \to 0.$$

Tensoring this sequence by $\mathscr{O}_X(-1)$ we find that $N_X(-1)$ is generated by sections from $H^0(X, T_{\mathbb{P}^N}|_X(-1)) \cong V^*$. Therefore, we obtain an imbedding $\mathbb{P}(N_X^*(1)) \subset X \times \mathbb{P}(V)$ and projection onto the second factor defines the duality map

$$\phi\colon \mathbb{P}(N_X^*(1)) \to \mathbb{P}^N$$

from the conormal variety $\mathbb{P}(N_X^*(1))$ to the dual projective space $\mathbb{P}^N = \mathbb{P}(V)$. The image of ϕ is defined to be the dual variety of X: $X' = \phi(X) \subset \mathbb{P}^N$. This construction is a true duality in the sense that $(X')' \cong X$. The dual variety is also known as the discriminant variety since it is isomorphic to the variety of singular (tangent) hyperplane sections. A simple dimension count shows that $\dim \mathbb{P}(N_X^*(1)) = N - 1$. For most smooth projective varieties the duality map is generically one-to-one and X' has codimension 1. The difference from this norm is called the defect.

3.1 **Definition.** The defect of a subvariety $X \subset \mathbb{P}^N$ is defined to be

$$\operatorname{def} X = N - \dim X' - 1.$$

In particular, $\operatorname{def} X$ is the dimension of a general fiber of the duality map ϕ. When the imbedding is defined by the sections of a particular very ample line bundle L we refer to the defect of X as the defect of (X, L) or of L and denote it by $\operatorname{def}(X, L)$.

If X is contained in a hyperplane H then X' is a cone over the dual variety of $X \subset H$ with vertex equal to the point dual to H. Conversely, if X' is a cone, then X is contained in the hyperplane dual to the vertex of the cone, see [6]. The defect remains the same whether we consider X as a subvariety of $\mathbb{P}^N$ or of $H \cong \mathbb{P}^{N-1}$. Projective space $\mathbb{P}^n$ with $L = \mathscr{O}_{\mathbb{P}^n}(1)$ is a special case. Since there are no singular hyperplane sections, the dual variety is empty. In order to be compatible with later formulas, we adopt the convention that $\operatorname{def}\mathbb{P}^n = n$ which by the above definition is the same as assigning the dimension -1 to the empty set.

3.2 **Proposition** [6, 7, 10]. *Let $X \subset \mathbb{P}^N$ be a smooth nonlinear projective variety. Let $n = \dim X$, $k = \operatorname{def} X$, and $n' = \dim X' = N - k - 1$.*

(1) *If Y is a smooth hyperplane section of X, then* $\operatorname{def} Y = \max\{0, k-1\}$.

(2) *The general fiber $F \subset \mathbb{P}(N_X^*(1))$ of ϕ over a point corresponding to a tangent hyperplane H is the singular locus of $X \cap H$ and is isomorphic to some linear projective subspace of dimension k: $F \cong \operatorname{Sing}(X \cap H) \cong \mathbb{P}^k$.*

(3) *If X is a curve, a surface, or a complete intersection, then $k = 0$.*

(4) *If $k > 0$ then $k \equiv n \bmod 2$.*

(5) *$n' \geq n$, and if X' is nonsingular then $n' = n$.*

(6) *If $k \geq n/2$, then X is a $\mathbb{P}^m$ bundle over a smooth projective variety where $m = (n+k)/2$ and the fibers are imbedded linearly.*

(7) *The betti numbers of X satisfy*

$$b_n = b_{n-2}, \qquad b_{n-1} = b_{n-3}, \dots, \qquad b_{n-k+1} = b_{n-k-1}.$$

The defect is related to the nef value in the following way.

3.3 **Proposition** [2, 0.12, 1.2], [3, 2.4, 3.1]. *Let X be a smooth projective variety and let L be a very ample line bundle on X. Assume the canonical bundle of X is not nef and let $\phi: X \to Y$ be the nef value morphism with general fiber Z. If* $\operatorname{def}(X, L) > 0$ *then*

$$\operatorname{def}(X, L) = \operatorname{def}(Z, L_Z) - \dim Y = 2(\tau(X, L) - 1) - \dim X$$

and $\operatorname{Pic} Z \cong \mathbb{Z}$. *Conversely, if* $\operatorname{def}(Z, L_Z) > 0$, *then* $\operatorname{def}(X, L) > 0$.

An important part of the classification of homogeneous spaces with positive defect involves products. We state here a general version of a formula for the defect of a product in terms of its factors. This proposition is a refinement of [2, 1.8].

3.4 **Proposition.** *Let X_1 and X_2 be smooth projective varieties with very ample line bundles L_1 and L_2, respectively, and* $\dim X_1 \geq \dim X_2$. *Let $X = X_1 \times X_2$ and $L = \operatorname{pr}_1^* L_1 \otimes \operatorname{pr}_2^* L_2$. Then* $\operatorname{def}(X, L) > 0$ *if and only if* $\operatorname{def}(X_1, L_1) > \dim X_2$. *When this is the case,* $\operatorname{def}(X, L) = \operatorname{def}(X_1, L_1) - \dim X_2 > 0$.

Proof. ($\Rightarrow$): Assume $\operatorname{def}(X, L) > 0$ and let $\tau = \tau(X, L)$. Since

$$K_X \otimes L^\tau = \operatorname{pr}_1^*(K_{X_1} \otimes L_1^\tau) \otimes \operatorname{pr}_2^*(K_{X_2} \otimes L_2^\tau)$$

both $K_{X_i} \otimes L_i^\tau$, $i = 1, 2$, are nef but not both can be ample.

Suppose $K_{X_i} \otimes L_i^\tau$, $i = 1, 2$, are both not ample. Then the nef values of all the varieties are equal, $\tau = \tau(X_1, L_1) = \tau(X_2, L_2)$. By Proposition 3.3, $\tau = (\dim X_1 + \dim X_2 + \operatorname{def} X)/2 + 1$, but since $\tau(X_i, L_i) \leq \dim X_i + 1$, $i = 1, 2$, we get

$$(\dim X_1 + \dim X_2 + \operatorname{def} X)/2 + 1 \leq \dim X_1 + 1, \dim X_2 + 1$$

which implies $\dim X_1 + \operatorname{def} X \leq \dim X_2$ and $\dim X_2 + \operatorname{def} X \leq \dim X_1$ so $\operatorname{def} X \leq 0$, contradicting our hypothesis. Therefore, we may assume that $K_{X_2} \otimes L_2^\tau$ is ample and $K_{X_1} \otimes L_1^\tau$ is not (the other possibility would ultimately contradict $\dim X_1 \geq \dim X_2$). In particular, $\tau = \tau(X_1, L_1)$. It follows that the nef value morphism $\psi: X \to Y$ must have the form $\psi = \psi_1 \times \mathrm{id}_{X_2}$ where $\psi_1: X_1 \to Y_1$ is the nef value morphism for X_1. Since the general fiber Z of ψ is isomorphic to the general fiber of ψ_1, and $L_Z = L_{1Z}$, we apply Proposition 3.3 to obtain

$$\operatorname{def} X = \operatorname{def} Z - \dim Y = \operatorname{def} Z - \dim Y_1 - \dim X_2 > 0$$

and $\operatorname{def} X_1 = \operatorname{def} Z - \dim Y_1 > \dim X_2$.

($\Leftarrow$): Assume $\operatorname{def} X_1 > \dim X_2$ and let $\tau_1 = \tau(X_1, L_1)$. By Proposition 3.3, $\tau_1 = (\dim X_1 + \operatorname{def} X_1)/2 + 1$. Since $\dim X_1 \geq \operatorname{def} X_1 > \dim X_2$, we obtain $\tau_1 > \dim X_2 + 1$. It follows that $K_{X_1} \otimes L_1^{\tau_1}$ is nef but not ample, and $K_{X_2} \otimes L_2^{\tau_1}$ is ample. Therefore, $K_X \otimes L^{\tau_1}$ is nef but not ample and $\tau = \tau_1$. As above, we find that $\psi = \psi_1 \times \mathrm{id}_{X_2}$, the general fiber Z of ψ is isomorphic to the general fiber of ψ_1, $L_Z = L_{1Z}$, and $Y = Y_1 \times X_2$. Again by Proposition 3.3, $\operatorname{def} X = \operatorname{def} Z - \dim Y = \operatorname{def} Z - \dim X_2 - \dim Y_1$, and $\operatorname{def} X_1 = \operatorname{def} Z - \dim Y_1$, hence $\operatorname{def} X = \operatorname{def} X_1 - \dim X_2 > 0$. □

These propositions yield the following statements about the defect of a homogeneous line bundle.

3.5 **Corollary.** *Let $X = G/P$ and let L be a line bundle on X. Let $X = X_1 \times \cdots \times X_s$ be the decomposition of X given in Proposition* 1.1.1.

(1) $\operatorname{def}(X, L) > 0$ *iff* $\operatorname{def}(X_i, L_i) > \operatorname{codim}_X X_i$ *for some* $1 \leq i \leq s$. *In this case* $\operatorname{def}(X, L) = \operatorname{def}(X_i, L_i) - \operatorname{codim}_X X_i > 0$.

(2) *If L is ample, the nef value morphism $\phi: X \to Y$ associated to L is a homogeneous fibration $G/P \to G/Q$ with fiber $Z = Q/P$ where Q is a parabolic subgroup of G and P is a maximal subgroup of Q. If* $\operatorname{def}(X, L) > 0$, *then* $\operatorname{def}(X, L) = \operatorname{def}(Z, L_Z) - \dim Y$.

This corollary shows that, up to products and fibrations, it is sufficient to classify homogeneous spaces with positive defect for the case where G is simple and P is maximal. The nef values computed in Corollary 2.4 show that very few of those spaces meet the numerical criterion of Proposition 3.3 for positive defect.

3.6 **Corollary.** *Let $X = G/P$ where G is a simple complex Lie group and P is a maximal parabolic subgroup. Let L be the ample generator of the line bundles on X. Let $\tau = \tau(X, L)$, $n = \dim X$, and $k = 2(\tau - 1) - n$. Then $k > 0$ iff X is one of the following:*

(1) $A_l/P_1 \cong A_l/P_l$ *(projective space),* $\tau = l + 1$, $n = k = l$,
(2) $A_l/P_2 \cong A_l/P_{l-1}$ *(Grassmann),* $n = 2(l-1)$, $\tau = l + 1$, $k = 2$,
(3) A_5/P_3 *(Grassmann),* $n = 9$, $\tau = 6$, $k = 1$,
(4) B_l/P_1 *(quadric),* $n = \tau = 2l - 1$, $k = 2l$,
(5) $B_2/P_2 = D_4/P_4$ *(projective space),* $n = 3$, $\tau = 4$, $k = 3$,
(6) $B_3/P_3 = D_4/P_4$ *(quadric),* $n = \tau = 6$, $k = 4$,
(7) $B_4/P_4 = D_5/P_5$ *(spinor),* $n = 10$, $\tau = 8$, $k = 4$,
(8) $B_5/P_5 = D_6/P_6$ *(spinor),* $n = 15$, $\tau = 10$, $k = 3$,
(9) $B_6/P_6 = D_7/P_6$ *(spinor),* $n = 21$, $\tau = 12$, $k = 1$,
(10) C_l/P_1 *(projective space),* $n = 2l - 1$, $\tau = 2l$, $k = 2l - 1$,
(11) C_l/P_2, $n = 4l - 5$, $\tau = 2l - 1$, $k = 1$,
(12) D_l/P_1 *(quadric),* $n = \tau = 2l - 2$, $k = 2l - 4$,
(13) $E_6/P_1 \cong E_6/P_5$, $n = 16$, $\tau = 12$, $k = 6$,

(14) E_7/P_1, $n = 27$, $\tau = 18$, $k = 7$,
(15) F_4/P_4, $n = 15$, $\tau = 11$, $k = 5$,
(16) G_2/P_1 *(quadric)*, $n = \tau = 5$, $k = 3$.

The listed value of k is the defect only if the defect is positive, which is not the case for most of the entries. For example, quadrics are hypersurfaces and obviously have defect 0 (they are self-dual). The fact that $A_l/P_2 = \mathrm{Gr}(2, l+1)$ has positive defect only when $l+1$ is odd has been known for some time [8, 12]. The 10-dimensional spinor variety $B_4/P_4 = D_5/P_5$ is also known to have defect 4, see e.g., [6, 13].

The space C_l/P_2 is easily seen to have defect 0. Recall that $C_l = Sp(2l)$ is the stabilizer of a generic 2-form in $\bigwedge^2 \mathbb{C}^{2l}$ and stabilizes a (complementary) hyperplane through a point of the form $[v_1 \wedge v_2] \in \mathbb{P}(\bigwedge^2 \mathbb{C}^{2l})$. The isotropy subgroup of $[v_1 \wedge v_2]$ is conjugate to P_2 and therefore C_l/P_2, which has dimension $4l-5$, is a hyperplane section in $\mathrm{Gr}(2, 2l)$. This Grassmann variety has defect 0 as we just pointed out, and so by Proposition 3.2.1 the defect of C_l/P_2 is also 0.

The exceptional variety E_6/P_1 cannot have positive defect because its betti numbers do not follow the pattern of Proposition 3.2.7. The odd betti numbers of E_6/P_1 are zero, and the pertinent even betti numbers are $b_{16} = 3$, $b_{14} = 2$, and $b_{12} = 2$, see, e.g., [14]. Therefore, the defect of E_6/P_1 is 0.

The remaining cases are the spinor varieties, D_6/P_6, D_7/P_7, and the exceptional varieties E_7/P_1, F_4/P_4. These spaces do not violate any of the simple criteria for positive defect. We shall show that they have defect 0 by computing the fiber dimension of the duality map $\phi\colon \mathbb{P}(N_X^*(1)) \to \mathbb{P}(V)$. This is a special case of the following general situation.

Let $E = G \times_P E_0$ be a homogeneous vector bundle over X. If E is spanned by global sections, then the evaluation map of sections $X \times V^* \to E$ is surjective, where $V^* = H^0(X, E)$. Therefore, the projectivization $\mathbb{P}(E^*)$ imbeds into $X \times \mathbb{P}(V)$ and projection onto the second factor yields an equivariant map $\phi\colon \mathbb{P}(E) \to \mathbb{P}(V)$ which imbeds each fiber $\mathbb{P}(E_0)$ linearly into $P(V)$. We let $\Lambda(E_0) \subset \Lambda(V)$ denote the weights (repeated according to their multiplicity) of a maximal torus on $E_0 \subset V$. Let R_X be the roots of X and define

$$\Xi = \Lambda(V) \cap (\Lambda(E_0) + R_X), \qquad \Theta = \Lambda(E_0) \cap (\Xi - R_X).$$

Let z_ν be a fixed weight vector in V of weight $\nu \in \Lambda(V)$, and let x_α be a root vector in $\mathfrak{u}$ for the root $\alpha \in R_X$. Then for $\mu \in \Lambda(V)$, $\alpha \in R_X$ we define constants M_α^μ by the equation $x_\alpha \cdot z_{\mu-\alpha} = M_\alpha^\mu z_\mu$ if $\mu - \alpha \in \Lambda(V)$ and 0 otherwise.

3.7 **Theorem.** *We retain the notation and assumptions of the previous paragraph. Let c be an arbitrary vector $[c_\nu]_{\nu\in\Theta}$ and let $M(c)$ be the matrix $[M_\alpha^\mu c_{\mu-\alpha}]$ $(\mu \in \Xi, \alpha \in R_X)$. Then the dimension k of a general fiber of $\phi\colon \mathbb{P}(E) \to \mathbb{P}(V)$ satisfies $k \le \dim X - \operatorname{rank} M(c)$.*

Proof. The fiber dimension of $\phi\colon \mathbb{P}(E) \to \mathbb{P}(V)$ is constant on G-orbits, so the dimension of a general fiber can be calculated over a generic point $[z] \in \mathbb{P}(E_0)$. Now

$$\begin{aligned} k &= \dim \phi^{-1}([z]) = \dim\{[g, [w]] \in G \times_P \mathbb{P}(E_0) | g \cdot [w] = [z]\} \\ &= \dim\{g \in G | g^{-1} \cdot z \in E_0\} - \dim P = \dim\{u \in U | u \cdot z \in N_0^*(1)\} \end{aligned}$$

where U is the unipotent subgroup of G generated by the root groups U_α, $\alpha \in R_X$. The last equality comes from the fact the dimension in question can be determined near the identity in G and any $g \in G$ near the identity can be factored uniquely as $g = u \cdot p$ with $u \in U$, $p \in P$.

Passing to the Lie algebra $\mathfrak{u}$ of U, we find that there exist vectors $x_1, \dots, x_k \in \mathfrak{u}$ such that $x_i \cdot z \in E_0$ and are linearly independent, $i = 1, \dots, k$. With respect to the fixed basis write $z = \sum_{\nu \in \Lambda(E_0)} c_\nu z_\nu \in E_0$ and $x = \sum_{\alpha \in R_X} a_\alpha x_\alpha \in \mathfrak{u}$. Then

$$x \cdot z = \sum_{\alpha \in R_X} \sum_{\nu \in \Lambda(E_0)} a_\alpha c_\nu x_\alpha \cdot z_\nu = \sum_{\mu \in \Lambda(V)} \left[\sum_{\substack{\alpha \in R_X, \nu \in \Lambda(E_0) \\ \nu + \alpha = \mu}} M_\alpha^\mu c_\nu a_\alpha \right] x_\mu .$$

Thus, $x \cdot z \in E_0$ if and only if for every $\mu \in \Xi$ we have $\sum_{\alpha \in R_X} M_\alpha^\mu c_{\mu - \alpha} a_\alpha = 0$. Since we have k independent solutions $[a_\alpha]_{\alpha \in R_X}$ to these equations, the rank of the matrix $[M_\alpha^\mu c_{\mu-\alpha}]$ must be $\leq \dim X - k$ as claimed. □

This theorem applied to the duality map can be used to verify the well-known examples of homogeneous spaces with positive defect. We shall only use it to show that the remaining cases from Corollary 3.6 do not have positive defect.

3.8 **Proposition.** *If X is D_6/P_6, D_7/P_7, E_7/P_1, or F_4/P_4 and L is the generator of the ample line bundles on X, then* $\operatorname{def}(X, L) = 0$.

Proof. Let $G = D_l$ and let $\lambda = \lambda_l$. Let $\varepsilon_1, \dots, \varepsilon_l$ be the standard orthonormal basis for the Lie algebra so that $\lambda = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$ and the roots of X are $R_X = \{\varepsilon_i + \varepsilon_j | 1 \leq i < j \leq l\}$. The representation V is the spinor representation whose weights are $\Lambda(V) = \{\frac{1}{2}(\sigma_1 \varepsilon_1, \dots, \sigma_l \varepsilon_l) | \sigma_i = \pm 1, \sigma_1 \cdots \sigma_l = 1\}$. Thus, a weight of V can be identified by the (even number of) coordinates in the ε-basis that are negative. Let $T_0^*(1)$ and $N_0^*(1)$ denote the fibers of $T_X^*(1)$ and $N_X^*(1)$ over the identity coset in X. We find that $\Lambda(T_0^*(1)) = \{\lambda - (\varepsilon_i + \varepsilon_j) | 1 \leq i < j \leq l\}$ so that these weights have exactly two negative coordinates and hence $\Lambda(N_0^*(1)) = \Lambda(V) \backslash (\Lambda(T_0^*(1)) \cup \{\lambda\}) = \{\frac{1}{2}(\sigma_1 \varepsilon_1 + \cdots + \sigma_l \varepsilon_l)\}$ where the number of $\sigma_i = -1$ is 4, 6, etc. It follows that

$$\begin{aligned} \Xi &= \{\mu_{pq} = \lambda - (\varepsilon_p + \varepsilon_q) | 1 \leq p < q \leq l\}, \\ \Theta &= \{\nu_{ijpq} = \lambda - (\varepsilon_i + \varepsilon_j + \varepsilon_p + \varepsilon_q) | 1 \leq i < j < p < q \leq l\}. \end{aligned}$$

Let x_{ij} (resp. y_{ij}) be a root vector in $\mathfrak{u}$ for the roots $\varepsilon_i + \varepsilon_j$ (resp. $-(\varepsilon_i + \varepsilon_j)$). We fix a basis for V consisting of the vector v_0 of weight λ, $v_{pq} = y_{pq} v_0$ of weight μ_{ij}, $1 \leq p < q \leq l$, $v_{ijpq} = y_{ij} y_{pq} v_0$ of weight ν_{ijpq}, $1 \leq i < j < p < q \leq l$, etc. Let $\langle ijpq \rangle$ denote the list of indices i, j, p, q rearranged to be in their natural order. If $\mu = \mu_{pq} \in \Xi$ and $\alpha = \varepsilon_i + \varepsilon_j \in R_X$, then we write M_{ij}^{pq} for M_α^μ so that $x_{ij} \cdot v_{\langle ijpq \rangle} = M_{ij}^{pq} v_{pq}$. Then

$$M_{ij}^{pq} = \begin{cases} 0, & i, j, p, q \text{ not pairwise disjoint}, \\ [[x_{ij}, y_{st}], y_{uv}]/y_{pq}, & stuv = \langle ijpq \rangle. \end{cases}$$

The brackets can be resolved using the standard matrix representation: $x_{ij} = e_{l+j,i} - e_{l+i,j}$ and $y_{ij} = e_{i,l+j} - e_{j,l+i}$ $1 \le i < j \le l$. Here e_{ij} is the usual elementary $2l \times 2l$-matrix. The resulting values for M_{ij}^{pq} are ± 1.

Direct calculation shows that when $l = 6$ or 7, the rank of the matrix $[M_{ij}^{pq} c_{ijpq}]$ is 15 or 21, respectively, for generic $c = [c_{ijpq}]_{1 \le i < j < p < q \le l}$. Therefore, by Corollary 3.6 and Theorem 3.7 the defects of D_6/P_6 and D_7/P_7 are both zero.

For the other two cases, we need only show that the rank of $[M_\alpha^\mu c_{\mu-\alpha}]$ is greater than the value predicted by Corollary 3.6 and Theorem 3.7 for some vector $c = [c_\nu]_{\nu \in \Theta}$ if the defect were positive. This estimate of the rank can actually be deduced from the simpler matrix $[c_{\mu-\alpha}]$ $(\mu \in \Xi, \alpha \in R_X)$ where we set $c_{\mu-\alpha} = 0$ if $\mu - \alpha \notin \Theta$.

The data for the case $X = F_4/P_4$, with the fundamental weights as a basis, is as follows.

$$\begin{aligned} R_X = \{&(0,0,-1,2), (0,-1,1,1), (-1,1,-1,1), (1,0,-1,1), \\ &(-1,0,1,0), (1,-1,1,0), (-1,0,0,2), (0,1,-1,0), \\ &(1,-1,0,2), (0,0,1,-1), (0,1,-2,2), (0,0,0,1), \\ &\qquad (0,-1,2,0), (-1,1,0,0), (1,0,0,0)\}, \end{aligned}$$

$$\begin{aligned} \Xi = \{&(0,0,1,-1), (0,1,-1,0), (1,-1,1,0), (-1,0,1,0), \\ &(1,0,-1,1), (-1,1,-1,1), (1,0,0,-1), (0,-1,1,1), \\ &(-1,1,0,-1), (0,0,-1,2), (0,-1,2,-1), (0,0,0,0), \\ &\qquad (0,1,-2,1), (1,-1,0,1), (-1,0,0,1)\}, \end{aligned}$$

$$\begin{aligned} \Theta = \{&(-1,0,1,-1), (-1,1,-1,0), (0,-1,1,0), (0,0,-1,1), \\ &(0,0,0,-1), (0,0,1,-2), (0,1,-1,-1), (1,-1,1,-1), \\ &\qquad (1,0,-1,0), (0,0,0,0)\}. \end{aligned}$$

Using the letters $a, b, c, \ldots$ to represent the constants c_ν, $\nu \in \Theta$, in order, we find that the matrix $[c_{\mu-\alpha}]$ is

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & j & 0 & f & g & h & a \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & j & 0 & 0 & f & g & 0 & i & b \\
0 & 0 & 0 & 0 & 0 & j & 0 & 0 & f & 0 & 0 & h & i & 0 & c \\
0 & 0 & 0 & 0 & j & 0 & f & 0 & 0 & 0 & 0 & a & b & c & 0 \\
0 & 0 & 0 & j & 0 & 0 & 0 & 0 & g & 0 & h & i & 0 & 0 & d \\
0 & 0 & j & 0 & 0 & 0 & g & 0 & 0 & 0 & a & b & 0 & d & 0 \\
0 & 0 & 0 & f & 0 & g & 0 & h & 0 & i & 0 & 0 & 0 & 0 & e \\
0 & j & 0 & 0 & 0 & 0 & h & 0 & a & 0 & 0 & c & d & 0 & 0 \\
0 & 0 & f & 0 & g & 0 & 0 & a & 0 & b & 0 & 0 & 0 & e & 0 \\
j & 0 & 0 & 0 & 0 & 0 & i & 0 & b & 0 & c & d & 0 & 0 & 0 \\
0 & f & 0 & 0 & h & a & 0 & 0 & 0 & c & 0 & 0 & e & 0 & 0 \\
f & g & h & a & i & b & 0 & c & 0 & d & 0 & e & 0 & 0 & 0 \\
g & 0 & i & b & 0 & 0 & 0 & d & 0 & 0 & e & 0 & 0 & 0 & 0 \\
h & i & 0 & c & 0 & d & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 \\
a & b & c & 0 & d & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

We now choose the vector $[c_\nu]$ so that all but the constants e and j equal 0. The resulting matrix is nonsingular. Moreover, it remains nonsingular if we change each of the j's and e's to other nonzero values. This proves that the matrix $[M_\alpha^\mu c_{\mu-\alpha}]$ is nonsingular for this choice of $[c_\nu]$. We therefore conclude that the defect of X is zero.

A similar calculation and argument shows that for the case $X = E_7/P_1$ the matrix $[M_\alpha^\mu c_{\mu-\alpha}]$ has rank $\geq \dim X - 2$, so again the defect of X is 0. □

We can conclude at this point that the only homogeneous spaces $X = G/P$, P maximal parabolic, that have positive defect are the familiar three examples: $\mathbb{P}^n$, $\mathrm{Gr}(2, 2m+1)$, and S_4.

4. Classifications

We now make the final conclusions about which homogeneous spaces $X = G/P$ have positive defect and show how the self-dual homogeneous spaces can be used to classify real homogeneous hypersurfaces in $\mathbb{P}^N$. We need one more technical fact.

4.1 **Lemma.** *Let G be a simple complex Lie group and let $P \subset Q$ be proper parabolic subgroups of G such that $Q/P \cong \mathbb{P}^k$. Then $\dim G/Q > k$.*

Proof. Let $\Psi \subset \Delta$ be the subset of simple roots defining Q. There must be a connected component of the Dynkin diagram of Ψ isomorphic to type a_k or type c_m where $m = (k+1)/2$, since these represent the only two simple groups that act transitively on projective space $Q/P \cong \mathbb{P}^k$. Let Ψ' denote the subset of simple roots corresponding to this connected component. Since G is simple, its Dynkin diagram is connected, and therefore there is a simple root $\beta \in \Delta\backslash\Psi'$ such that $\Phi' = \Psi' \cup \{\beta\}$ forms a set of simple roots for a simple subgroup $G' \subset G$. Let $Q' = G' \cap Q$ and $P' = G' \cap P$. By construction, Q' is a maximal parabolic subgroup of G', $Q'/P' \cong \mathbb{P}^k$, and $\dim G'/Q' \leq \dim G/Q$. The proof is complete if we show $k < \dim G'/Q'$.

Therefore, we may assume from the outset that Q is a maximal parabolic subgroup of a simple group G defined by a connected subset of simple roots Ψ of type a_k or c_m, $m = (k+1)/2$. In particular, Ψ is either type a_k and G has rank $k+1$, or Ψ is type c_m and $G = C_{m+1}$, $m = (k+1)/2$, or $G = F_4$, $m = 3, k = 5$. It is quite easy to check each of the simple types for these conditions. The smallest possibility for $\dim G/Q$ occurs for the case $G = A_{k+1}$ where $G/Q \cong \mathbb{P}^{k+1}$. □

4.2 **Theorem** (cf. [11]). *Let $X = G/P \subset \mathbb{P}^N$ be a homogeneous space and let L be an ample line bundle on X. If $k = \mathrm{def}(X, L) > 0$, then X is one of the following:*

(1) *A linear projective space $\mathbb{P}^n$, $k = n$.*

(2) *The Plücker imbedding of the Grassmann variety $\mathrm{Gr}(2, 2m+1)$, $k = 2$.*

(3) *The 10-dimensional spinor variety S_4 in $\mathbb{P}^{15}$, $k = 4$.*

(4) *$X_1 \times X_2$ where X_1 is one of the varieties in 1–3 and $\mathrm{def}\, X = \mathrm{def}\, X_1 - \dim X_2 > 0$.*

Proof. If X is not isomorphic to a product then we may assume G is simple, see Proposition 1.1.1. Also, whenever P is a maximal parabolic subgroup of G and $\mathrm{def}(X, L) > 0$, it follows immediately from [7, Theorem 1.3(b)] that L must be the generator of the ample line bundles on X.

Consider the nef value morphism $\phi\colon X = G/P \to Y = G/Q$ of Corollary 3.5.2. The fiber $Z = Q/P$ is isomorphic to the quotient of a simple group by a maximal parabolic subgroup. Since $\operatorname{def} Z > 0$, the only possibilities for such quotients are the three listed varieties as was demonstrated in §3. If $Z = \mathrm{Gr}(2, 2m+1)$ then $\dim Y < \operatorname{def} Z = 2$. If $\dim Y = 1$ then $Y \cong \mathbb{P}^1$ and $G = SL(2, \mathbb{C})$ which is too small to provide a nontrivial fibration. Therefore, $\dim Y = 0$ and $X = \mathrm{Gr}(2, 2m+1)$.

Similarly, if Z is S_4, then $\dim Y < \operatorname{def} Z = 4$. If Y is not a point, then Y must be $\mathbb{P}^r$, $r = 1, 2$, or 3, the 3-dimensional flag manifold, or a 3-dimensional quadric. The corresponding groups are A_r, $r = 1, 2$, or 3, and B_2. None of these groups contains either of the two complex simple Lie groups that acts transitively on the 10-dimensional spinor variety (B_4 or D_5). So again we conclude that the fibration is trivial and $X = S_4$.

The final possibility of a nontrivial fibration with $Z = \mathbb{P}^k$ can also be ruled out because of the condition $\dim Y < \operatorname{def} Z$ and Lemma 4.1. Therefore, if X is not a product, it must be one of the three listed varieties. If X is isomorphic to a product, then Corollary 3.5.1 implies statement 4. □

The following theorem holds for nonlinear smooth projective varieties $X \subset \mathbb{P}^N$ such that $\dim X = \dim X' \le \frac{2}{3}N$, see [6]. The version we present here for homogeneous spaces is simply a corollary of the above classification. This list also classifies those nonlinear homogeneous spaces with nonsingular dual varieties, see Proposition 3.2.5. If X is a linear projective space $X = \mathbb{P}^n \subset \mathbb{P}^N$ then the tangent hyperplanes are clearly parameterized by a complementary projective space $X' = \mathbb{P}^{N-n-1}$.

4.3 **Corollary.** *Let $X = G/P$ be a nonlinear homogeneous space imbedded in $\mathbb{P}^N$ by the sections of an ample line bundle L on X. If $\dim X = \dim X'$ then X is one of the following:*

(1) *A quadric hypersurface in $\mathbb{P}^{n+1}$.*
(2) *The Segre imbedding of $\mathbb{P}^{n-1} \times \mathbb{P}^1$ in $\mathbb{P}^{2n-1}$.*
(3) *The Plücker imbedding of $\mathrm{Gr}(2, 5)$ in $\mathbb{P}^9$.*
(4) *The 10-dimensional spinor variety S_4 in $\mathbb{P}^{15}$.*

Proof. The listed varieties are well known to be self-dual, see e.g., [6, 13]. If $X = \mathrm{Gr}(2, 2m+1)$ then $\dim X = 2(2m-1)$ and $\dim X' = m(2m+1) - 4$ and these are equal only when $m = 2$. By Theorem 4.2 it remains to check the case $X = X_1 \times X_2$. By Proposition 1.1.2, $L = \mathrm{pr}_1^* L_1 \otimes \mathrm{pr}_2^* L_2$, so that $H^0(X, L) \cong H^0(X_1, L_1) \otimes H^0(X_2, L_2)$. Therefore, the imbedding dimension of X satisfies $N+1 = (N_1+1)(N_2+1)$ where N_i is the imbedding dimension of X_i under L_i, $i = 1, 2$. We know that $\operatorname{def} X = N - \dim X - 1 = \operatorname{def} X_1 - \dim X_2$. Hence, $N_1 N_2 + N_1 + N_2 = \dim X_1 + \operatorname{def} X_1 + 1$. Since $N_i \ge \dim X_i$ this equation becomes $\dim X_2(\dim X_1 + 1) \le \operatorname{def} X_1 + 1$. Now, X_1 must be a projective space, for otherwise $\operatorname{def} X_1 + 1 \le 5$ and this would imply that $\dim X_1 = 1$. Therefore, $(X_1, L_1) \cong (\mathbb{P}^{n_1}, \mathcal{O}_{\mathbb{P}^{n_1}}(1))$ and the previous equation yields $N_2 = 1$. This implies that $(X_2, L_2) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, and $X = \mathbb{P}^{n-1} \times \mathbb{P}^1$ as claimed. □

Certain real hypersurfaces in complex projective space are tubes over complex submanifolds, see [5]. This fact along with the above classification of self-dual homogeneous spaces can be used to classify homogeneous real hypersurfaces in complex projective space.

4.4 **Corollary** [1, 15]. *Let M be a homogeneous complete real hypersurface imbedded equivariantly in $\mathbb{P}^N$. Then M is a tube over a linear projective space or one of the 4 self-dual homogeneous spaces $X \subset \mathbb{P}^N$ listed in Corollary 4.3.*

Proof. Let $M = K/L$ where K is a compact Lie group and let ξ denote the normal vector field to M. If J denotes the complex structure operator, then $W = -J\xi$ is a tangent vector field. Because the imbedding is equivariant, W is left invariant under K and therefore its integral curves are given by 1-parameter subgroups of K and are geodesics. By [5], M is a tube over a complex submanifold $X \subset \mathbb{P}^N$ (a focal submanifold). In particular, X is homogeneous, $X = G/P$, and G acts transitively on the normal directions to X. It follows that the conormal variety $\mathbb{P}(N_X^*(1))$ itself is a homogeneous space, G/P_0, and therefore the image of $\phi\colon \mathbb{P}(N_X^*(1)) = G/P_0 \to X'$ must also be homogeneous, $X' = G/P'$. If X is not a linear projective space then $\dim X' = \dim X$, see Proposition 3.2.5, and so X is one of 4 self-dual homogeneous spaces by Corollary 4.3. □

Conversely, the classification of homogeneous real hypersurfaces [15] can also be used to obtain the list of self-dual homogeneous spaces. For, if a homogeneous space $X = G/P$ is self-dual, then by a symmetry argument, the conormal variety must be a homogeneous space under G. This implies that G acts transitively on the normal directions to X and hence a maximal compact subgroup of G must have a hypersurface orbit in the normal bundle of X. The resulting orbit is a homogeneous real hypersurface in $\mathbb{P}^N$ realized as a tube over X and therefore must be on the list given in [15], see also [1].

References

1. J. Berndt, *Real hypersurfaces with constant principal curvatures in complex space forms*, Geometry and Topology of Submanifolds. II (Avignon, 1988), World Scientific, Teaneck, N.J., 1990.
2. M. Beltrametti, M. Fania, and A. Sommese, *On the discriminant variety of a projective manifold*, Forum Math. **4** (1992), 529–547.
3. M. Beltrametti, A. Sommese, and J. Wiśniewski, *Results on varieties with many lines and their applications to adjunction theory*, Lecture Notes in Math., vol. 1507, Springer, Berlin, Heidelberg, and New York, 1992.
4. A. Borel, *Linear algebraic groups*, Benjamin, New York, 1971.
5. T. Cecil and P. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.
6. L. Ein, *Varieties with small dual varieties*. I, Invent. Math. **86** (1986), 63–74.
7. ——, *Varieties with small dual varieties*. II, Duke Math. J. **52** (1985), 895–907.
8. P. Griffiths and J. Harris, *Algebraic geometry and local differential geometry*, Ann. Sci. École Norm. Sup. **12** (1979), 355–432.
9. J. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, Berlin, Heidelberg, and New York, 1972.
10. S. Kleiman, *Tangency and duality*, Canad. Math. Soc. Conf. Proc., vol. 6, Amer. Math. Soc., Providence, R.I., 1986, pp. 164–225.
11. F. Knop and G. Menzel, *Dualen Varietäten von Fahnenvarietäten*, Comment. Math. Helv. **62** (1987), 38–61.
12. A. Lascoux, *Degree of the dual of a Grassmann variety*, Comm. Algebra **9(11)** (1981), 1215–1225.

13. R. Lazarsfeld and A. Van de Ven, *Topics in the geometry of projective space*, Recent Work of F. L. Zak, Birkhäuser, Basel, Boston, and Stuttgart, 1984.

14. D. Snow, *Vanishing theorems on compact hermitian symmetric spaces*, Math. Z. **198** (1988), 1–20.

15. R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495–506.

16. J. Tits, *Tabellen zu den Einfachen Lie Gruppen und ihre Darstellungen*, Lecture Notes in Math., vol. 40, Springer-Verlag, Berlin, Heidelberg, and New York, 1967.

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